2 Distributions

Exercise 2.1. Let $\Omega \subseteq \mathbb{R}^d$ be an open set and let $\{K_n\}_{n\in\mathbb{N}} \subset \Omega$ be a sequence of compact sets for which the following holds

$$K_n \subseteq \mathrm{int}(K_{n+1}) \quad \forall\, n \in \mathbb{N} \qquad \text{and} \qquad \bigcup_{n\in\mathbb{N}} K_n = \Omega,$$

and $K_0 = \varnothing$. For all $m \in \mathbb{N}$ and $n \in \mathbb{N}$, denote by $\|\cdot\|_{m,n} : , \mathscr{D}(\Omega) \to \mathbb{R}$ the function

$$\|\varphi\|_{m,n} = \sup_{|\alpha|\leq m} \|D^\alpha \varphi\|_{\mathrm{L}^\infty(\Omega\setminus K_n)}$$

Show that $\|\cdot\|_{m,n}$ is a semi-norm on $\mathscr{D}(\Omega)$.

Exercise 2.2. Let $\Omega \subseteq \mathbb{R}^d$ be an open set, $\varphi \in \mathscr{D}(\Omega)$ be a function and $\boldsymbol{\varepsilon} = \{\varepsilon_n\}_{n\in\mathbb{N}} \subset (0,\infty)$, $\boldsymbol{m} = \{m_n\}_{n\in\mathbb{N}} \subset \mathbb{N}$. Show that the following collection of sets

$$\mathcal{D}_{\varphi,\boldsymbol{\varepsilon},\boldsymbol{m}} = \mathscr{D}(\Omega) \cap \left\{ \psi : \|\psi - \varphi\|_{\boldsymbol{\varepsilon},\boldsymbol{m}} = \sup_{n\in\mathbb{N}} \left(\frac{1}{\varepsilon_n} \sup_{|\alpha|\leq m_n} \|D^\alpha \psi - D^\alpha \varphi\|_{\mathrm{L}^\infty(\Omega\setminus K_n)} \right) \leq 1 \right\}$$

for different choices of $\varphi, \boldsymbol{\varepsilon}, \boldsymbol{m}$, is a basis generating a topology on $\mathscr{D}(\Omega)$.

Exercise 2.3. Show that the topology introduced in Exercise 2.2 induces the following notion of convergence in $\mathscr{D}(\Omega)$.

> *A sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathscr{D}(\Omega)$ converges to $\varphi \in \mathscr{D}(\Omega)$ if and only if there exists a compact set $K \subset \Omega$ such that spt $\varphi_n \subseteq K$ for all $n \in \mathbb{N}$ and all φ_n and their derivatives of any order converge uniformly to φ and its respective derivative.*

Exercise 2.4. We recall the notation $\mathcal{D}(\Omega) = C_c^\infty(\Omega)$ and $\mathcal{E}(\Omega) = C^\infty(\Omega)$ and the notion of convergence

$$\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi \text{ in } \mathcal{E}(\Omega) \qquad \Longleftrightarrow \qquad \begin{array}{l} \forall\, K \subset \Omega \text{ compact}, \forall \alpha \in \mathbb{N}^d, \text{ we have} \\ \|D^\alpha \varphi_n - D^\alpha \varphi\|_{\mathrm{L}^\infty(K)} \to 0. \end{array}$$

Let T be a distribution in $\mathcal{D}'(\Omega)$ that does not have compact support. Show that it is possible to find a sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(\Omega)$ and some $\varphi \in \mathcal{E}(\Omega)$ such that $\varphi_n \to \varphi$ in the sense of $\mathcal{E}(\Omega)$ but $T(\varphi_n) \underset{n\to\infty}{\longrightarrow} \infty$.

Exercise 2.5. Compute the following distributions

$$\frac{d^2}{dt^2}\left[\big(H(t) - H(t-2)\big)(t^2 - t - 2)\right] \quad \text{and} \quad \frac{1-\cos(2\pi t)}{t} \sum_{k\in\mathbb{Z}} \delta'(t-k).$$